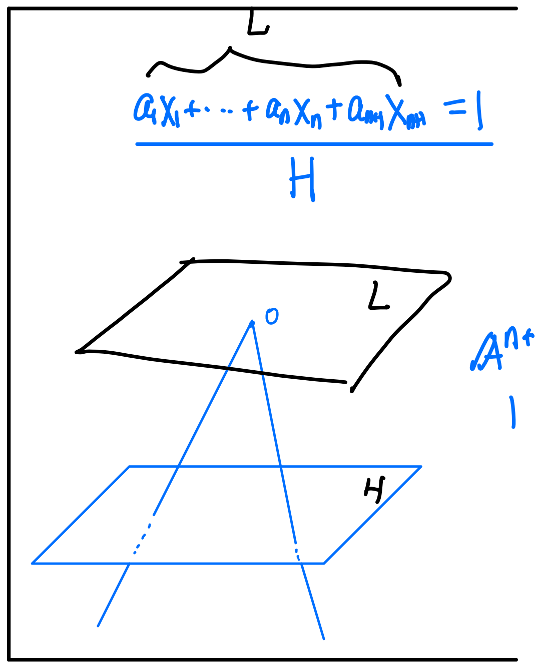
$$\{\text{aff. alg. sets in } \mathbb{A}^n\} \underset{(\cdot)_*}{\overset{(\cdot)^*}{\rightleftarrows}} \{\text{proj. alg. sets in } \mathbb{P}^n\}$$

$$W^* := V_p(I(W)^*) \subseteq \mathbb{P}^n$$

$$V_* := V_a(I(V)_*) \subseteq \mathbb{A}^n.$$

- Basic property of $(\cdot)^*$ & $(\cdot)_*$ $\Rightarrow$ $\mathbb{A}^n \xrightarrow{\sim} U_{n+1} \overset{\varphi_{n+1}}{\hookrightarrow} \mathbb{P}^n$.
- Similarly $\mathbb{A}^n \xrightarrow{\sim} U_i \overset{\varphi_i}{\hookrightarrow} \mathbb{P}^n$
- $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$ open covering.



general: $U_H := \{\ell \in \mathbb{P}^n \mid \ell \cap H \neq \phi\}$

$$= \mathbb{P}(\mathbb{A}^{n+1}) - \mathbb{P}(L)$$

Fact: 1) $V_a(F)^* = V_p(F^*)$

2). $\{\text{affine var.}\} \overset{1:1}{\longleftrightarrow} \{\text{proj. var not contained in } H_\infty\}$.

3). $\alpha : k(V^*) \xrightarrow{\cong} k(V)$ $\quad f/g \mapsto f_*/g_*$

$\alpha : \mathcal{O}_p(V^*) \cong \mathcal{O}_p(V)$.

- $\Gamma_h(V^*)_d \xrightarrow{(\cdot)_*} \Gamma(V)$ $\quad F \bmod I_p(V^*) \longmapsto F_* \bmod I$

$$\Rightarrow \alpha\left(\frac{F \bmod I(V^*)}{G \bmod I(V^*)}\right) := \frac{F_* \bmod I}{G_* \bmod I}$$

Example: $V = \mathbb{A}^1$, $V^* = \mathbb{P}^1$

$$\Gamma(V) = k[x] \quad \left(x \sim \frac{X}{Y}\right) \qquad \Gamma_h(V^*) = k[X,Y]$$

$$\frac{F(X,Y)}{G(X,Y)} = \frac{F(X,Y)/Y^d}{G(X,Y)/Y^d} = \frac{F(\frac{X}{Y},1)}{G(\frac{X}{Y},1)} \sim \frac{F_*(x)}{G_*(x)}$$

where F & G forms of the same degree.

	$\mathbb{A}^n$	$\mathbb{P}^n$	$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$
pt	$(x_1, \cdots, x_n)$	$[x_1 : \cdots : x_{n+1}]$	$(P_1, \cdots, P_r, P)$ $\quad P_i \in \mathbb{P}^{n_i}$, $P \in \mathbb{A}^n$
ring	$k[X_1, \cdots, X_n]$	$k[X_1, \cdots, X_{n+1}]$	$k[X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2}, \cdots, X_{rn_r} \times Y_1, \cdots, Y_m]$ $=: k[X_1, X_2, \cdots, X_r, Y]$
zero pt	$F(P)=0$	$F(P)=0$	$F(P) = 0 \overset{def}{\iff} \cdots$
alg set	$V(S)$	$V_p(S)$	$V_m(S) =: \{$
Ideal	$I(X)$	$I_p(X)$	$I_b(X) = \cdots$

$I_p(X)$ = homog. ($\Leftrightarrow$ generated by homog. poly.)

$I_b(X)$ = multi-homog. ($\Leftrightarrow$ generated by multi-homog.)

Def.: 1) F = multiform of multi deg $(p_1, \cdots, p_r, q)$ if F is a form of deg p_i when consider as in $k[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots X_r, Y][X_i]$. $\forall i$.

2) $I \triangleleft k[X_1, \cdots, X_r, Y]$ is called multihomogeneous, if I is generated by multiforms.

$V \subseteq \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$ Var. (irr)

$\Gamma_m(V) := k[X_1, \cdots, X_r, Y] / I_m(V)$

$\hookrightarrow$ multihomogeneous coordinate ring

$k_m(V) := \mathrm{Frac}(\Gamma_m(V))$

$k(V) = \{ z = \frac{f}{g} \in k_m(V) \mid f, g = \text{multiforms of the same bideg} \}$

$\mathcal{O}_P(V) := \{ z = \frac{f}{g} \in k(V) \mid g(P) \neq 0 \}$

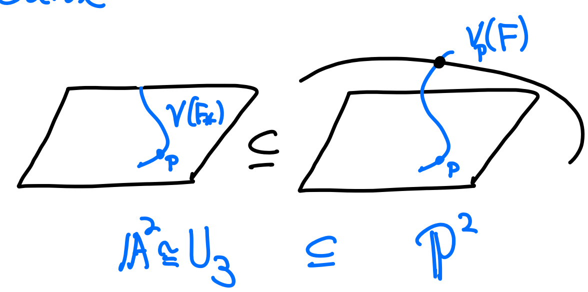
Chapter 5 Projective Plane Curve

$\forall\ F$ = Form in $k[X,Y,Z] \Rightarrow V(F)$ = hypersurface in $\mathbb{P}^2$

projective plane curve

$F_* \in k[X,Y]$.

(dehomogenize w.r.t. Z)



$\forall\ P \in V(F)_*$ (if not, dehomognize w.r.t X or Y)

$$m_P(F) := m_P(F_*)$$

- F = irr. $\quad P$ = simple $\Leftrightarrow m_P(F)=1 \Rightarrow \mathcal{O}_P(F)$ = DVR

$$\Rightarrow \mathrm{ord}_P^F : k(F) \to \mathbb{Z}$$

form $G \in k[X,Y,Z]$, $\quad G_* \in \mathcal{O}_P(\mathbb{P}^2)$ (5.1.3)

$$\overline{G_*} = G_* \ (\mathrm{mod}\ F) \in \mathcal{O}_P(F)$$

$\mathrm{ord}_P^F(G) := \mathrm{ord}_P^F(\overline{G_*})$ = order at P of G/H for any form H of same deg as G with $H(P) \neq 0$

$$I(P, F\cap G) := I(P, F_* \cap G_*)$$

§ 5.2. linear systems of curves

aim : moduli of curves of deg. d.

$\{M_1, \cdots, M_N\}$ = set of monomials in X,Y,Z of deg d.

$N = \frac{1}{2}(d+1)(d+2)$ $\quad N-1 = \frac{d(d+3)}{2}$

$$\mathbb{P}^{N-1} \xleftrightarrow{1:1} \{\text{curves of deg. } d\}$$

$$[a_1 : \cdots : a_N] \longmapsto F = a_1 M_1 + \cdots + a_N M_N$$

↳ well-defined.

λF, F stand for the same curve

Fact : the curves of deg. d form a proj. space of dim. $\frac{d(d+3)}{2}$.

Example : (1) $\{\text{line in } \mathbb{P}^2\} \xrightarrow{\sim} \mathbb{P}^2$

(2) $\{\text{conic in } \mathbb{P}^2\} \xrightarrow{\sim} \mathbb{P}^5$

(3) $\{\text{cubic in } \mathbb{P}^2\} \xrightarrow{\sim} \mathbb{P}^9$

(4) $\{\text{quartics in } \mathbb{P}^2\} \xrightarrow{\sim} \mathbb{P}^{14}$

$\vdots$

linear system of plane curves := a set of curves of degree d which forms a linear subvariety in $\mathbb{P}^{d(d+3)/2}$

$H^0(\mathcal{O}(d))$

Lemma: (1) $P \in \mathbb{P}^2$.

$$\{ C : \text{curve of deg. } d \mid P \in C \}$$

forms a hyperplane in $\mathbb{P}^{d(d+3)/2}$

(2) Give a set $S \subseteq \mathbb{P}^2$.

$$\{ F : \text{curve of deg } d \mid S \subseteq F \}$$

forms a linear subvariety of $\mathbb{P}^{d(d+3)/2}$

Pf: 1) $P \in F_{[a_1 : \cdots : a_N]} = \sum_i a_i M_i \iff \sum_i a_i M_i(P) = 0 \Rightarrow \checkmark$